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A Periodic Version of Lie Series for Normal Mode Dynamics

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Abstract: Lie series solutions of smooth dynamical systems are adapted for the classof periodic motions by invoking the temporal mode shapes of the classicimpact oscillator.

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Despite the fact that perfectly periodic motions never indeed happen, the class of periodic solutions of the differential equations of motion appears to be very important even when dealing with the chaotic dynamics [1]. Typical problem formulations and practical reasons for considering the periodic solutions can be also found in reference [2].

Let us consider a multiple degrees of freedom dynamical system described by the differential equations of motion with respect to the coordinate and velocity vectors

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -f(x, v, t), \\ \dot{t} &= 1, \end{aligned} \tag{1}$$

where the vector-function f is assumed to have as many derivatives as needed in a physically reasonable domain of the variables. Then, the dynamics of system (1) can be locally described by the Lie series [3]

$$x = \exp[(t - t_0)G]x_0 \equiv \left[1 + (t - t_0)G + \frac{1}{2!}(t - t_0)^2 G^2 + \dots\right] x_0, \tag{2}$$

$$G = v_0 \cdot \frac{\partial}{\partial x_0} - f(x_0, v_0, t_0) \cdot \frac{\partial}{\partial v_0} + \frac{\partial}{\partial t_0} \tag{3}$$

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where G is Lie operator associated with system (1), and $\{x_0, v_0, t_0\}$ is some initial point in the system' phase space.

Series (2) is simply Taylor series whose coefficients are calculated by enforcing equations (1). Unfortunately, this general idea is still of little use for oscillatory processes probably due to locality of expansion (2). In other words, even entire expansion (2) does not explicitly reveal such global characteristics of oscillations as their amplitude and period. Moreover, the corresponding truncated series produce increasingly growing errors as the time t runs away from the selected initial point t_0. In order to overcome these disadvantages, it is suggested to adapt the Lie series solution for the class of periodic motions as follows.

Theorem 1 *Assume that system (1) admits a periodic solution $x(t)$ of the period $T = 4a$ so that $x(t + 4a) = x(t)$ for any t, and some point $\{x_0, v_0, t_0\}$ belongs to this solution. Then such a solution can be expressed in the form*

$$x = \exp(aG)\{\cosh[a(\tau - 1)G] + e\sinh[a(\tau - 1)G]\}x_0, \tag{4}$$

where τ and e are triangular sine and rectangular cosine, whose periods are normalized to four and amplitudes are normalized to unity as

$$\tau(\varphi) = (2/\pi)\arcsin\sin(\pi\varphi/2) \tag{5}$$

and,

$$e(\varphi) = \operatorname{sgn}\cos(\pi\varphi/2) \tag{6}$$

respectively, and $\varphi = (t - t_0)/a$ is a re-scaled time. If, in addition, the solution is odd with respect to one half of the period, $x(t + 2a) = -x(t)$, then expression (4) simplifies to

$$\begin{aligned} x &= [\sinh(a\tau G) + e\cosh(a\tau G)]x_0 \\ &\equiv \left[a\tau G + \frac{1}{3!}(a\tau G)^3 + \cdots\right]x_0 + e\left[1 + \frac{1}{2!}(a\tau G)^2 + \cdots\right]x_0 \end{aligned} \tag{7}$$

Proof of expression (4) is obtained by substituting the identity [4]

$$\varphi = 1 + [\tau(\varphi) - 1]e(\varphi) \quad (-1 < \varphi < 3) \tag{8}$$

in (2) and taking into account that

$$e^2 = 1 \tag{9}$$

at almost every time instance[1]. In order to prove the particular case (7), one should keep in mind that $\exp(2aG)x_0 = x(t_0 + 2a) = -x_0$, as it follows from (2), and the oddness condition assumed.

Note that τ and e are indeed quite simple piece-wise linear functions; the above analytical expressions (5) and (6) just define them in the unit-form which enables one to avoid conditioning of computation in the original temporal scale, $t_0 \le t < \infty$. This possibility becomes essential when the dynamics includes some evolutionary component.

Physical meaning of relationship (8) is that, during the whole period, the time variable φ is expressed through the coordinate τ and velocity e of a classic particle freely oscillating

[1] The set of isolated points $\{\varphi\colon\ \tau(\varphi) = \pm 1\}$ appears to have no effect on the results [4].

between the two absolutely stiff barriers with no energy loss. Due to (9), this relationship possesses the algebraic structure of "hyperbolic complex numbers" as revealed by (4).

Let us outline possible applications of expressions (4) and (7). For the sake of simplicity, consider the particular case (7). Of course, formal expression (7) does not guarantee the existence of periodic solutions. In case some periodic solution does exist, one should be able to find the corresponding vectors x_0 and v_0 from appropriate conditions. In autonomous case, the scalar parameter, a, is also unknown and must be determined.

The related conditions are formulated as a requirement of smoothness of expression (7), which is generally non-smooth or even discontinuous due to the presence of non-smooth and discontinuous functions τ and e, respectively. The "smoothing" relations are obtained by eliminating the step-wise discontinuities of the coordinate and velocity vectors imposing the constraints

$$\begin{aligned} \cosh(aG)x_0 &= 0, \\ \cosh(aG)v_0 &= 0. \end{aligned} \tag{10}$$

In autonomous case, algebraic equations (10) represent a nonlinear eigenvalue problem, where a is an eigenvalue, and $\{x_0, v_0\}$ is a combined (state) eigenvector.

By narrowing the class of periodic motions to those on which the system passes its trajectory twice in the configurations space during the same period, one obtains a subclass of normal mode motions. For more physically meaningful definitions and discussions, see reference [5]. Let us formulate the corresponding problem based on the periodic Lie series solutions.

Consider the vibrating system

$$\ddot{x} + f(x) = 0, \quad x \in R^n, \tag{11}$$

where $f(-x) = -f(x)$, and the initial conditions are $x|_{t=0} = x_0 = 0$ and $\dot{x}|_{t=0} = v_0$.

The normal mode solutions of system (11) are obtained as a particular case of (7) and (10)

$$x = \sinh(a\tau G)x_0|_{x_0=0}, \tag{12}$$

$$\cosh(aG)v_0|_{x_0=0}, \tag{13}$$

where the initial vector $x_0 = 0$ is substituted into the expressions only after all degrees of the differential operator

$$G = v_0 \cdot \frac{\partial}{\partial x_0} - f(x_0) \cdot \frac{\partial}{\partial v_0}$$

have been applied.

Relationship (12) can be interpreted as a parametric equation of normal mode trajectories of the system with the parameter interval $-1 \le \tau \le 1$.

Let us illustrate relationships (12) and (13) based on the linear system so that the result could be compared with the well known conventional solution.

Example 1 Suppose that $f(x) = Kx$, where K is positively defined symmetric $n \times n$-matrix with eigen-system $\{v_0, \omega^2\}$ so that $Kv_0 = \omega^2 v_0$. In this case, by applying the operator G twice, one obtains that v_0 is also an eigenvector of the operator G^2, namely,

$G^2 v_0 = -\omega^2 v_0$. Then, keeping in mind the power series form of expressions (12) and (13) as those in (7) and sequentially applying the operator G^2, gives $x = (v_0/\omega)\sin(a\omega\tau)$ and $\cos(a\omega\tau) = 0$, respectively. Notably, the last equation shows that there exist an infinite number of roots $\{a\}$ related to the same eigenfrequency ω! However, it is easily to find that all the roots produce the same solution in terms of the original time t. The minimal quarter of the period is $a = \pi/(2\omega)$, therefore $x = (v_0/\omega)\sin(\pi\tau/2)$, and $\tau = (2/\pi)\arcsin\sin\omega t$.

Nonlinear cases and the related problems dealing with truncated expansions of (13) will be further discussed in a full-length paper.

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